Introduction to Hydrodynamics I

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Contents

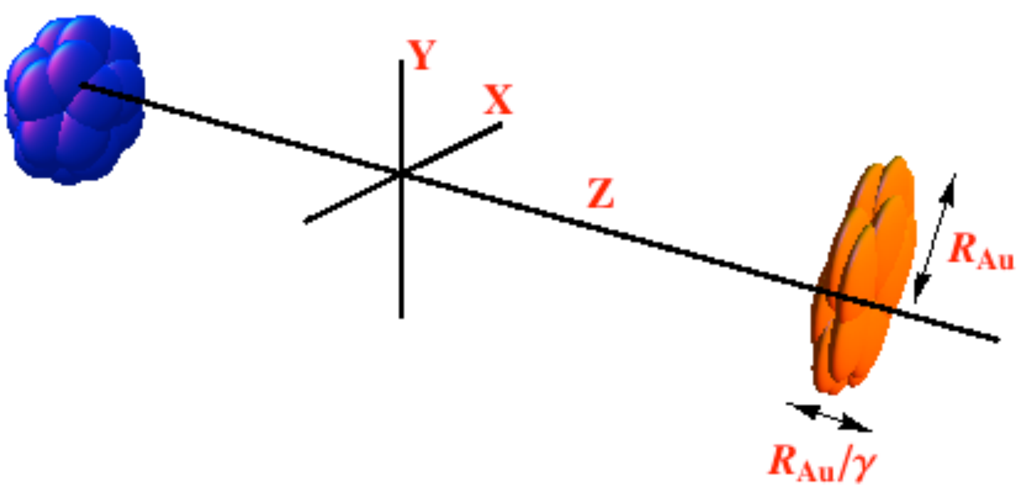
Space Time Picture of Heavy Ion Collisions

1. A heavy ion collision at RHIC has a center of mass energy

$$\sqrt{s} = 200 \frac{\text{GeV}}{\text{nucleon pair}}$$

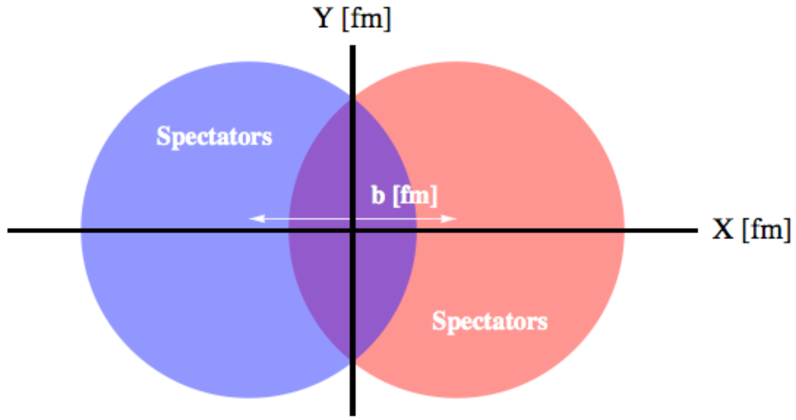
2. Each projectile has a "gamma factor" of

$$\gamma \approx E/m_N \approx 100$$

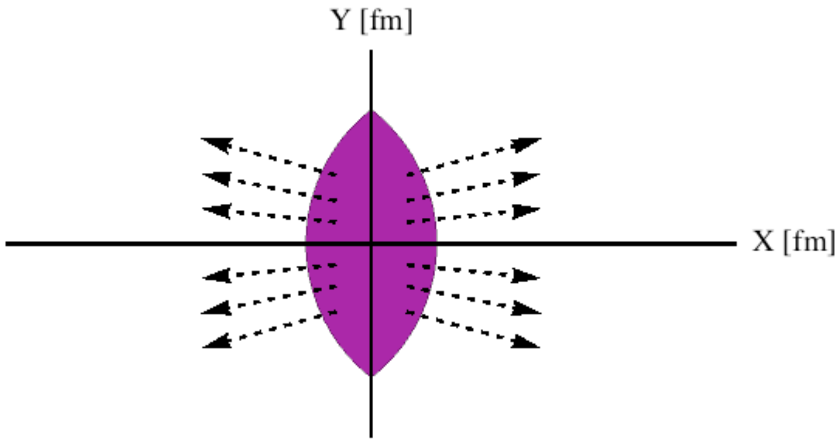


Phenomenon of Elliptic Flow

1. In an off-central collision there is an initial anisotropy in the transverse plane of excited nuclear matter



2. Pressure gradients drive flow preferentially in x-direction

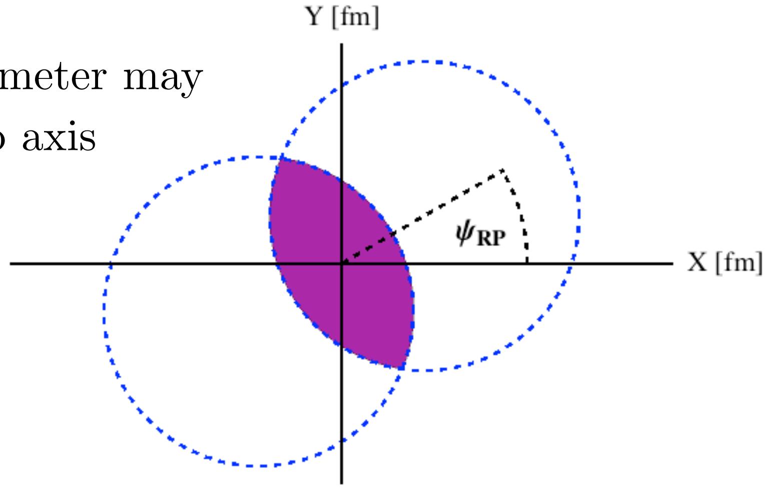


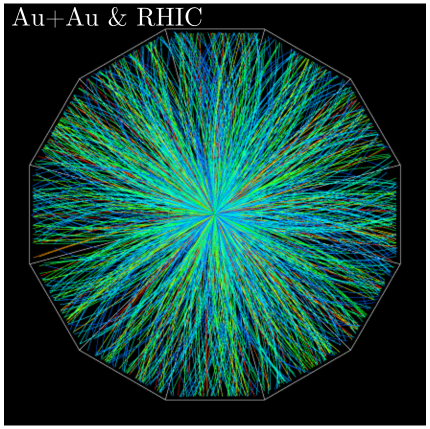
3. This is quantified by expanding the yields in terms of a Fourier series

$$\frac{1}{p_\perp}\frac{dN}{dy\,dp_\perp d\phi_p} = \frac{1}{2\pi p_\perp}\frac{dN}{dy\,dp_\perp}\left(1 + 2v_2(p_\perp)\cos 2(\phi - 2\Psi_{RP}) + \cdots\right)$$

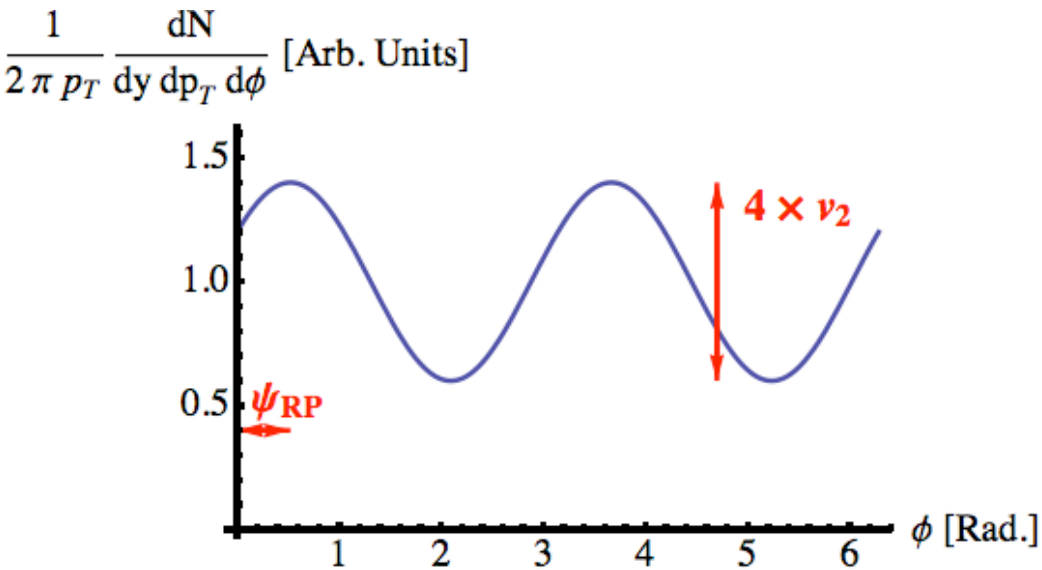
Measurement of Elliptic Flow

1. In general impact parameter may not be aligned with lab axis





2. The simplest method is the "event plane method"



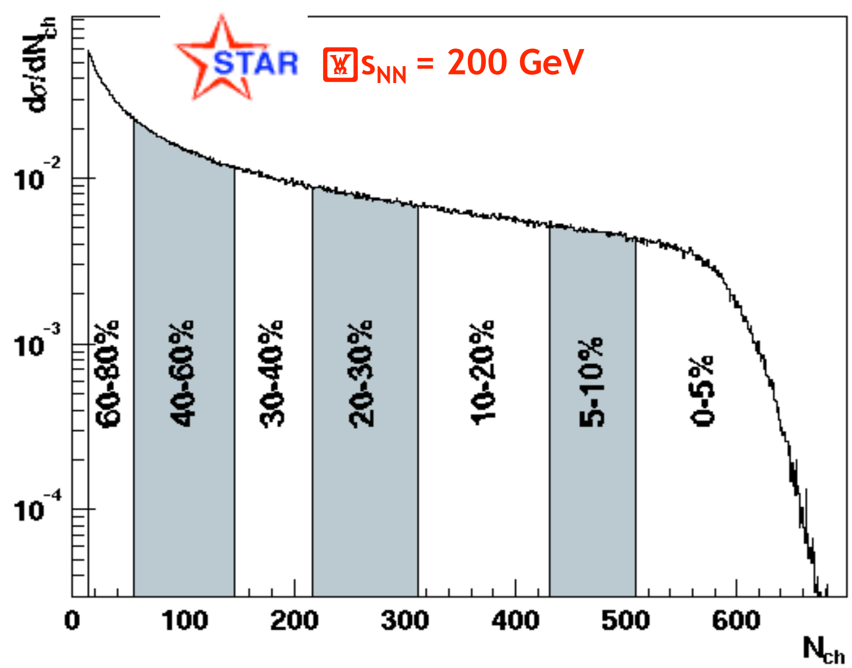
Measurement of Centrality

1. We can choose an impact parameter by selecting events with a certain multiplicity

2. At high energies the cross section is almost geometrical $\sigma_{tot} \approx \pi(2R_A)^2$

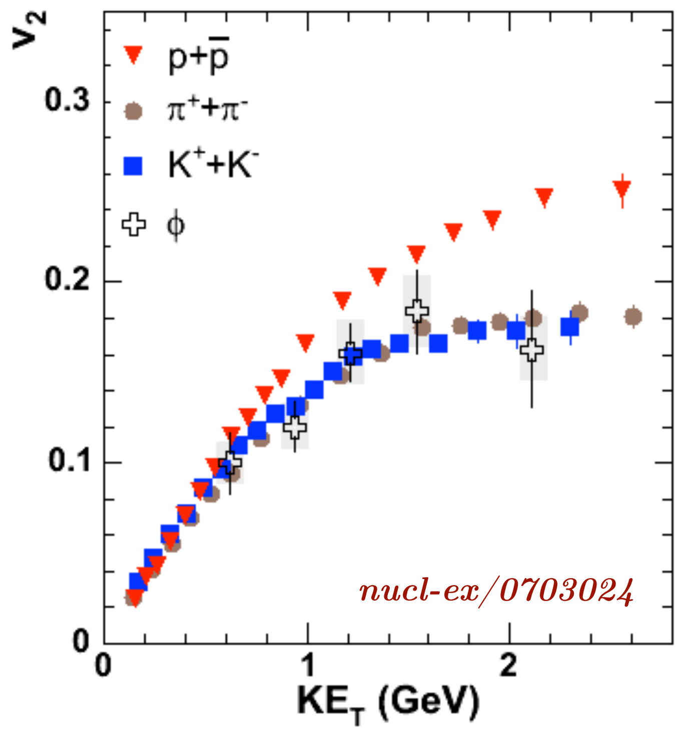
3. If we assume that the events with the highest multiplicity have the smallest impact parameter

$$\left(\frac{b}{2R_a}\right)^2 \approx \%\ \text{Centrality}$$



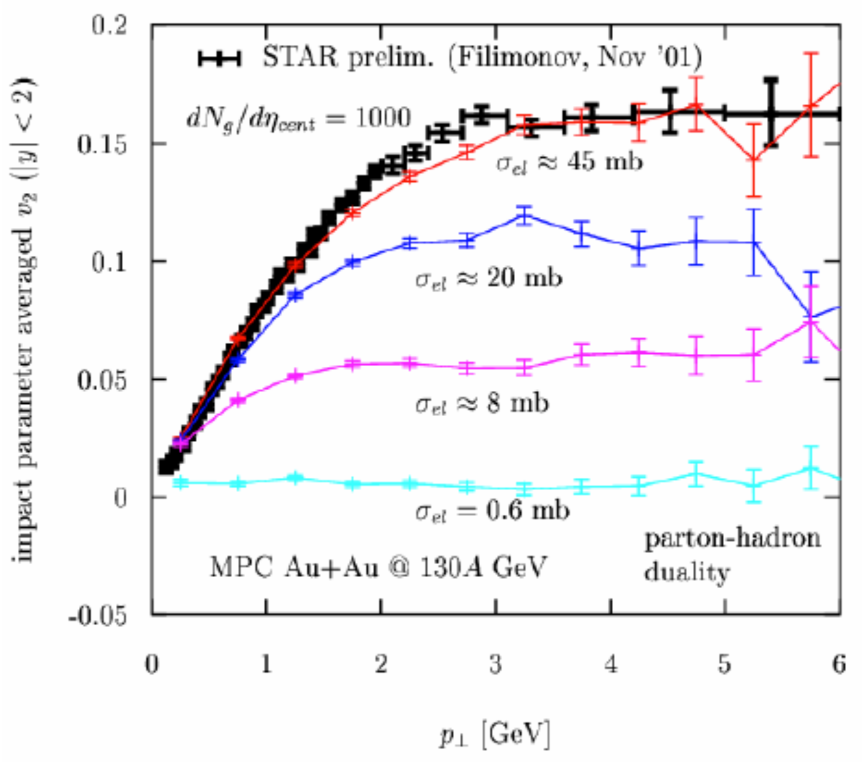
Measured Elliptic Flow

1. Such a large elliptic flow is rather surprising since the production processes in pQCD knows nothing about the geometry



Elliptic flow from MPC

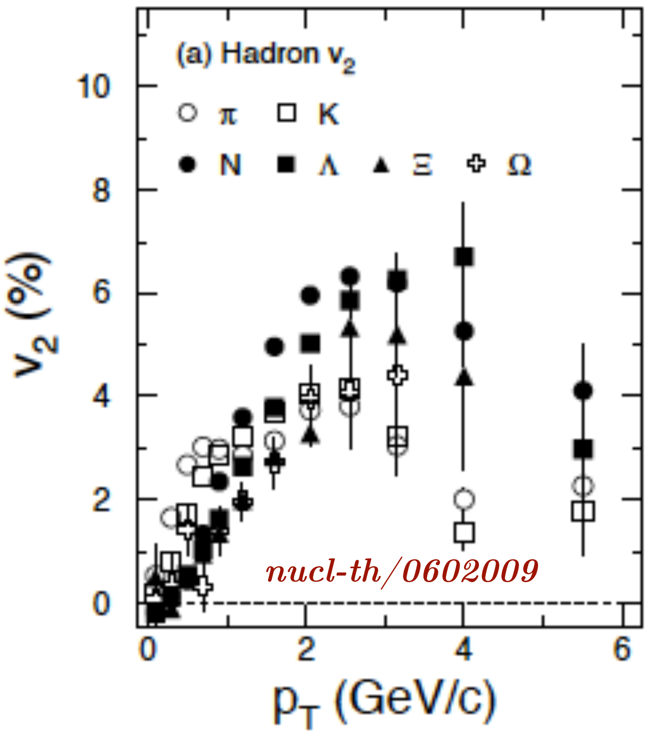
1. If re-scattering among the produced gluons is included the pQCD cross section is too small in order to get the observed flow



$\approx 15 \times$ pQCD

Elliptic flow in UrQMD

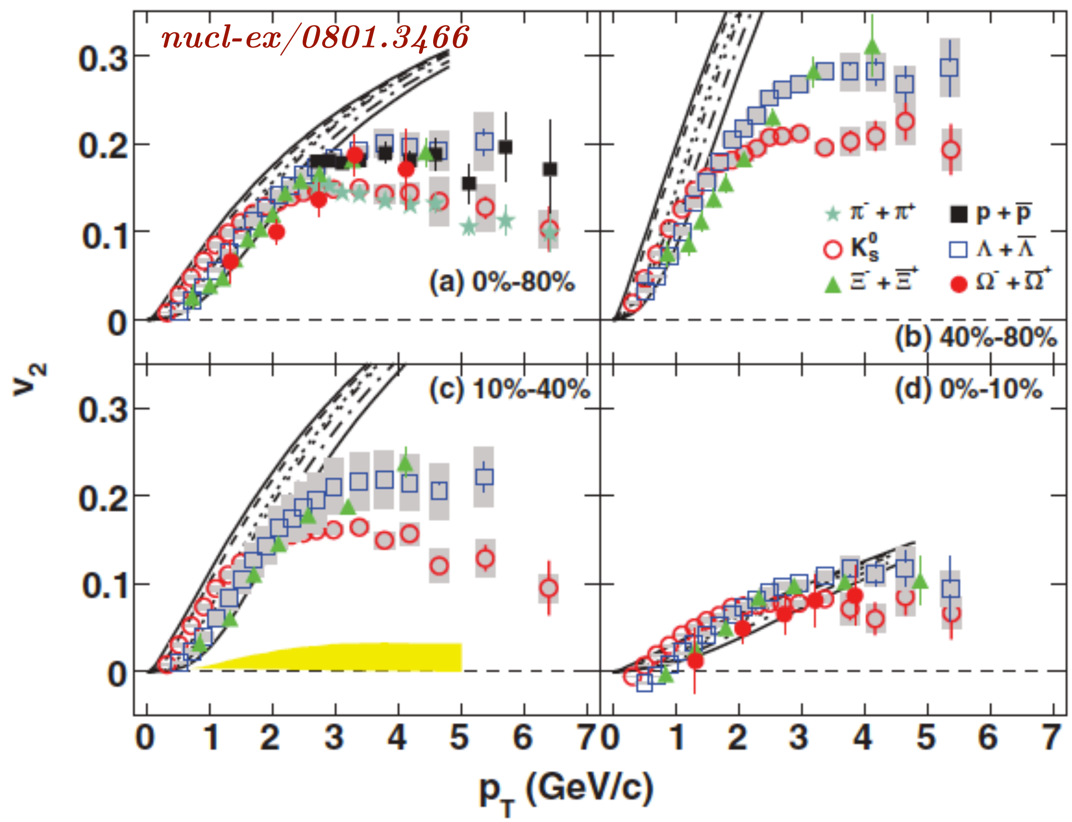
1. Flow is clearly present from hadronic re-scattering



2. But the magnitude is off by a factor ≈4

Elliptic flow from Ideal Hydrodynamics

1. The theoretical limit of zero mean free path is able to reproduce observed flow



Fluid Dynamics from Kinetic Theory

1. Starting point of our kinetic description is the quasi-particle phase space density

$$f(\mathbf{p}, \mathbf{x}, t) d^3x d^3p$$

2. Liouville Theorem (collision-less system)

$$\frac{p^\mu \partial_\mu}{E_\mathbf{p}} f(\mathbf{p}, \mathbf{x}, t) \equiv \left(\partial_t + v^i_\mathbf{p} \partial_i\right) f(\mathbf{p}, \mathbf{x}, t) = 0$$

$$v^i_\mathbf{p} \equiv \frac{p^i}{E_\mathbf{p}}$$

3. Or in the presence of collisions

$$\left(\partial_t + v^i_\mathbf{p} \partial_i\right) f(\mathbf{p}, \mathbf{x}, t) = -\mathcal{C}[f, \mathbf{p}]$$

Notation

1. Number density

$$n(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3} f(\mathbf{p},\mathbf{x},t) \equiv \int_{\mathbf{p}} E_{\mathbf{p}} f(\mathbf{p},\mathbf{x},t)$$

$$\int_{\mathbf{p}} \equiv \int \frac{d^3p}{(2\pi)^3 E_{\mathbf{p}}}$$

2. And since we are only considering on-shell excitations

$$\int \frac{d^3p}{(2\pi)^3 E_{\mathbf{p}}} \equiv 2\int d^4p \; \theta(p^0)\delta(p^\mu p_\mu - m^2)$$

Particle Flow

1. Density: $$n(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} f(\mathbf{p}, \mathbf{x}, t)$$

2. Current: $$j^i(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} v^i_{\mathbf{p}} f(\mathbf{p}, \mathbf{x}, t) \qquad v^i_{\mathbf{p}} \equiv \frac{p^i}{E_{\mathbf{p}}}$$

3. can be grouped into the four-current:

$$j^\mu(\mathbf{x}, t) = \left(n, j^i\right) = \int_{\mathbf{p}} p^\mu f(\mathbf{p}, \mathbf{x}, t)$$

Energy Flow

1. Energy density: $$e(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} f(\mathbf{p},\mathbf{x},t)$$

2. Energy flow: $$\int \frac{d^3p}{(2\pi)^3} p^i v^j_{\mathbf{p}} f(\mathbf{p},\mathbf{x},t)$$

3. Momentum flow: $$\int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} v^i_{\mathbf{p}} f(\mathbf{p},\mathbf{x},t)$$

4. can be grouped into the energy momentum tensor:

$$T^{\mu\nu} = \int_{\mathbf{p}} p^\mu p^\nu f(\mathbf{p},\mathbf{x},t)$$

Summary

1. So far everything has been very general
   - The only assumption being that we have well-defined quasi-particles

2. We have simply written the definition of the particle and momentum flow in compact notation

$$j^\mu(\mathbf{x},t) = \left(n, j^i\right) = \int_{\mathbf{p}} p^\mu f(\mathbf{p},\mathbf{x},t)$$

$$T^{\mu\nu} = \int_{\mathbf{p}} p^\mu p^\nu f(\mathbf{p},\mathbf{x},t)$$

3. Next step is to include collisions ...

Boltzmann Equation

1. Boltzmann equation is a specific choice of collision operator

$$\left(\partial_t + v^i_{\mathbf{p}}\partial_i\right) f(\mathbf{p}, \mathbf{x}, t) = -\mathcal{C}[f, \mathbf{p}]$$

$$\mathcal{C}[f, \mathbf{p}] = \frac{1}{p}\int_{\mathbf{q}}\int_{\mathbf{q}'}\int_{\mathbf{p}'} |\mathcal{M}|^2 (2\pi)^4 \delta^4\left(P + Q - P' - Q'\right)\left[f_{\mathbf{q}'} f_{\mathbf{p}'} - f_{\mathbf{q}} f_{\mathbf{p}}\right]$$

2. Collisional Invariants:

   For any quantity χ conserved during the collision

$$\int \frac{d^3p}{(2\pi)^3}\, \chi\, \mathcal{C}[f, \mathbf{p}] = 0$$

Macroscopic EoM

1. Starting with Boltzmann equation

$$\left(\partial_t + v^i_{\mathbf{p}}\partial_i\right) f(\mathbf{p},\mathbf{x},t) = -\mathcal{C}[f,\mathbf{p}]$$

2. Act on both sides with $\displaystyle\int_{\mathbf{p}} \chi$

$$\int_{\mathbf{p}} \chi \left(\partial_t + v^i_{\mathbf{p}}\partial_i\right) f(\mathbf{p},\mathbf{x},t) = -\int_{\mathbf{p}} \chi\mathcal{C}[f,\mathbf{p}] = 0$$

3. Pulling out the derivatives

$$\partial_t \int_{\mathbf{p}} \chi f + \partial_i \int_{\mathbf{p}} v^i_{\mathbf{p}} \chi f = 0$$

Macroscopic EoM

1. Current Conservation:

$$\chi = E_{\mathbf{p}} : \quad \partial_\mu j^\mu(X) \equiv \partial_t n(\mathbf{x}, t) + \partial_i j^i(\mathbf{x}, t) = 0$$

2. Energy Momentum Conservation

$$\chi = E_{\mathbf{p}} p^\mu : \quad \partial_\mu T^{\mu\nu}(X) \equiv \partial_t T^{0\nu}(\mathbf{x}, t) + \partial_i T^{i\nu}(\mathbf{x}, t) = 0$$

$$\nu = 0, 1, 2, 3$$

3. Still pretty trivial
   - Used ene-mtm conservation on microscopic level and got ene-mtm conservation on macroscopic EoM

Flow velocity

1. Up to now we haven't discussed the hydrodynamic flow velocity

2. Not to be confused with an individual particle's momentum

$$v^i_{\mathbf{p}} \equiv \frac{p^i}{E_{\mathbf{p}}}$$

General properties of flow velocity

1. Time length vector of length c=1

$$u^\mu u_\mu = -1$$

2. An any space-time point can find a LRF such that

$$u^\mu_{LRF} = (1,0,0,0)$$

3. It will be useful to define a projector

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$$

4. which removes component of vector parallel to velocity

$$\Delta^{\mu\nu} u_\mu = 0$$

Eckart Definition

1. Let's tie the hydrodynamic velocity to the flow of particles

$$u^{\nu} = \frac{-j^{\nu}}{u^{\alpha} j_{\alpha}}$$

2. Eckart's definition implies that in the LRF

$$j^{\mu}_{LRF} = (n, 0, 0, 0)$$

3. Interpretation: flow velocity is the average particle velocity

4. Not suitable for relativistic systems since particle number is not conserved

Landau-Lifshitz Definition

1. Instead we can tie the velocity to the flow of momentum

$$u_\nu = -\frac{T^\mu_\nu u_\mu}{u_\alpha T^{\alpha\beta} u_\beta}$$

2. This is the "Landau-Lifshitz" definition and we shall use this convention

3. This is the frame where there is no flow of momentum

$$T^{0i}_{LRF} = T^{i0}_{LRF} = 0$$

Coming back to Boltzmann Eqn.

1. We want to solve the Boltzmann equation in the limit of small

$$\epsilon \equiv \frac{l}{L}$$

2. Without being too precise

$$\left(\partial_t + v^i_{\mathbf{p}}\partial_i\right) f(\mathbf{p}, \mathbf{x}, t) = -\mathcal{C}[f, \mathbf{p}]$$

$$\frac{u}{L} f \quad \sim \quad v_c f$$

$$v_c = n\langle \sigma v_{12} \rangle$$

3. or in operator notation $\quad \mathcal{L} f \;=\; \dfrac{1}{\epsilon}\mathcal{C}[f, \mathbf{p}]$

Zeroth order solution

1. Attempt a series expansion of $\mathcal{L}f \;=\; \frac{1}{\epsilon}\mathcal{C}[f,\mathbf{p}]$

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots$$

2. Then at zeroth order $\mathcal{C}[f_0,\mathbf{p}] = 0$

3. Looking at the form of the collision operator

$$\mathcal{C}[f,\mathbf{p}] = \frac{1}{p}\int_{\mathbf{q}}\int_{\mathbf{q}'}\int_{\mathbf{p}'} |\mathcal{M}|^2(2\pi)^4\delta^4\left(P+Q-P'-Q'\right)\left[f_{\mathbf{q}'}f_{\mathbf{p}'} - f_{\mathbf{q}}f_{\mathbf{p}}\right]$$

we must have $f^0_{\mathbf{q}'}f^0_{\mathbf{p}'} = f^0_{\mathbf{q}}f^0_{\mathbf{p}}$ at zeroth order.

Zeroth order solution (cont.)

1. Equilibrium distribution must satisfy $f^0_{\mathbf{q}'} f^0_{\mathbf{p}'} = f^0_{\mathbf{q}} f^0_{\mathbf{p}}$

2. and therefore $\ln f^0_{\mathbf{q}'} + \ln f^0_{\mathbf{p}'} = \ln f^0_{\mathbf{q}} + \ln f^0_{\mathbf{p}}$

3. so f_0 must be a linear combination of collisional invariants

$$\ln f^0(P, X) = a(X) + b_\mu(X) p^\mu$$

4. Exercise: Show f_0 must take the following form

$$f^0(P, X) = \exp\left(\frac{p^\mu u_\mu - \mu}{T}\right)$$

5. Notice this form didn't depend on details of matrix element

Euler Equations

1. Exercise:

   Using the definition of the current and Stress-Energy Tensor

   $$T_0^{\mu\nu} = \int_{\mathbf{p}} p^\mu p^\nu f_0(\mathbf{p}, \mathbf{x}, t) \qquad j_0^\mu(\mathbf{x}, t) = \int_{\mathbf{p}} p^\mu f_0(\mathbf{p}, \mathbf{x}, t)$$

   Show that:

   $$T_0^{\mu\nu} = \epsilon u^\mu u^\nu + p\Delta^{\mu\nu} \qquad j_0^\mu = n u^\mu$$

2. The ideal stress-energy tensor along with the EoMs

   $$\partial_\mu T_0^{\mu\nu}(X) = 0 \qquad \partial_\mu j_0^\mu(X) = 0$$

   are known as the relativistic Euler Equations.

Euler Equations (cont.)

1. We derived the Euler Equations from first principles (although under very restrictive approximations)

$$T_0^{\mu\nu} = e u^\mu u^\nu + p \Delta^{\mu\nu} \qquad\qquad \partial_\mu T_0^{\mu\nu}(X) = 0$$

Exercise: Rewrite the Euler equations in the following form

$$\begin{aligned} De &= -\left(e+p\right)\nabla_\mu u^\mu \\ Du^\mu &= -\frac{\nabla^\mu p}{e+p} \end{aligned}$$

where $D \equiv u^\mu \partial_\mu$ and $\nabla^\mu = \Delta^{\mu\nu}\partial_\nu$.

τ-η coordinates

1. Want to use variables more suitable for HIC

$$\tau \equiv \sqrt{t^2 - z^2} \qquad t \equiv \tau \cosh \eta$$

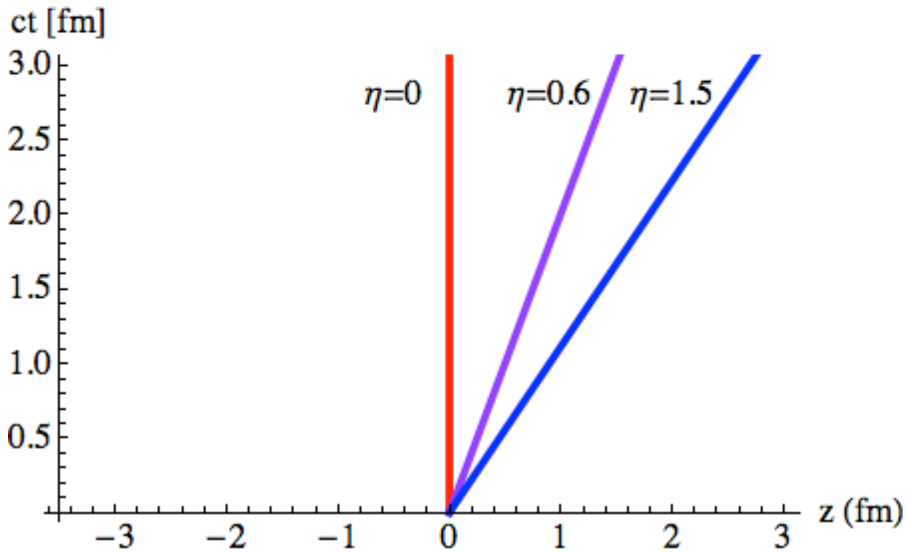
$$\eta \equiv \frac{1}{2} \ln \frac{t+z}{t-z} \qquad z \equiv \tau \sinh \eta$$

2. Coordinates transverse to beam direction remain the same

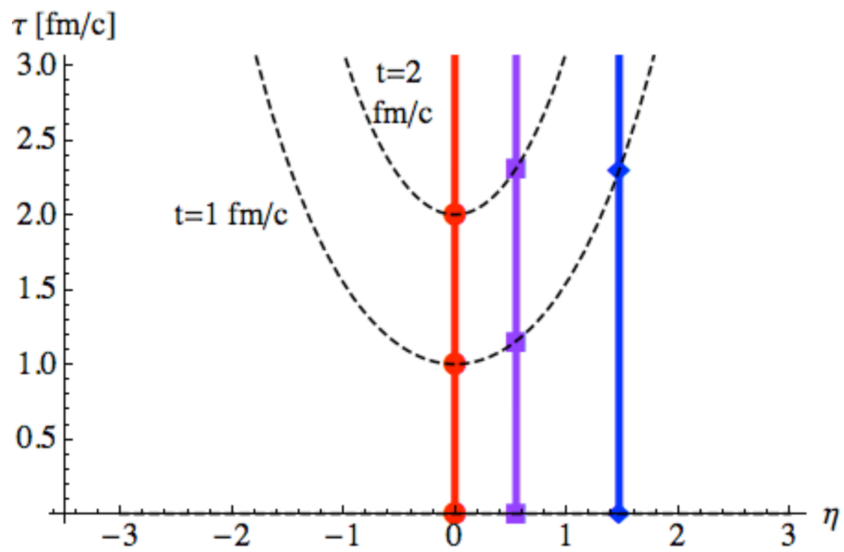
3. Exercise: show

$$d^4x \equiv dt d^3x = \tau d\tau d\eta d^2 x_\perp$$

Simple Example of τ-η coordinates

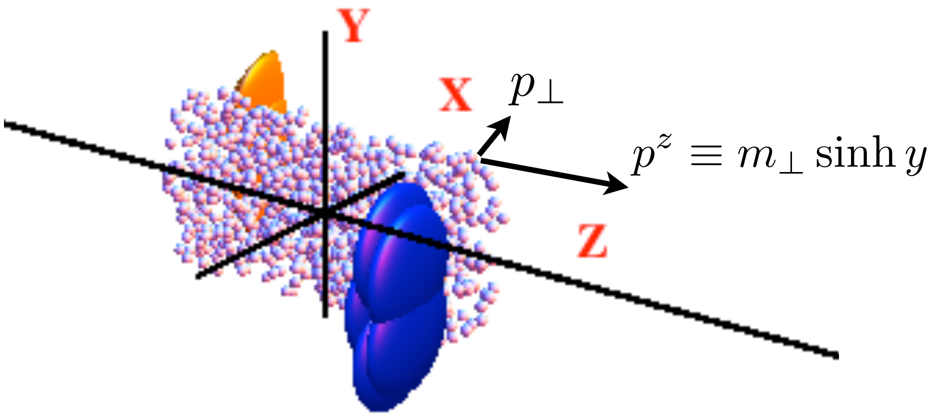


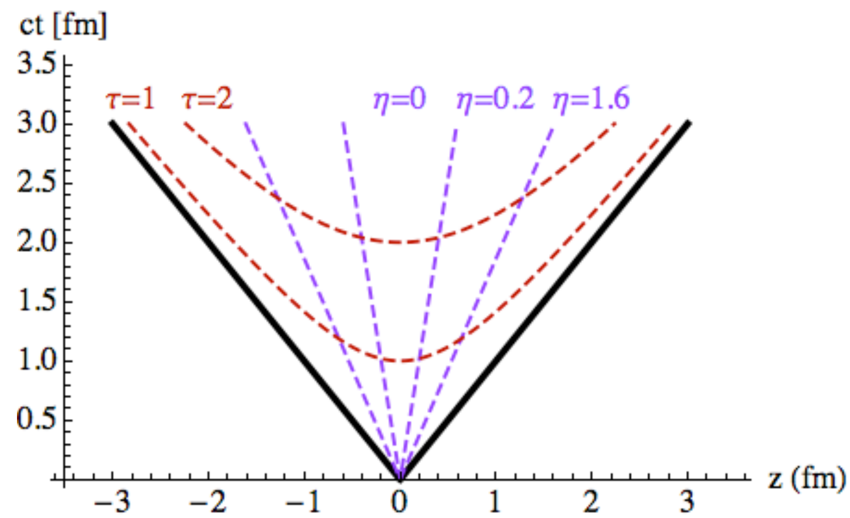
Consider three particles moving in z direction at 0c, 0.5c and 0.9c



In τ-η these particles sit at constant η.

Space Time Picture of Heavy Ion Collisions





1. In a high energy collision there is a separation of scales between the longitudinal and transverse momentum

2. This leads to a correlation between momentum and position

$$\frac{p^z}{E_{\mathbf{p}}} \sim \frac{z}{t} \qquad \text{or} \qquad y \sim \eta$$

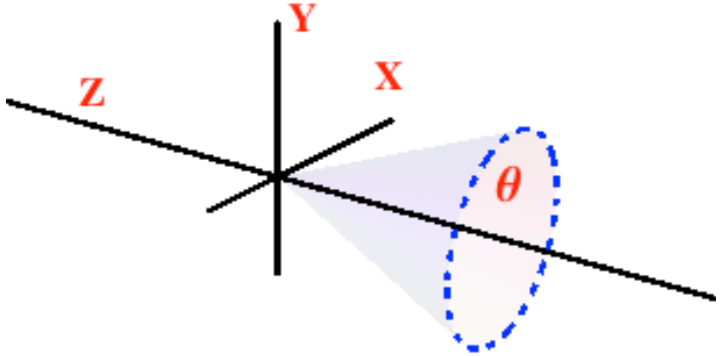
The three rapidities

1. Spatial Rapidity $$\eta \equiv \frac{1}{2} \ln \frac{t+z}{t-z}$$

2. Particle Rapidity $$y \equiv \frac{1}{2} \ln \frac{E+p_z}{E-p_z}$$

3. Particle Pseudo-rapidity

$$\eta_{\text{pseudo}} \equiv \frac{1}{2} \ln \left(\frac{|\vec{p}| + p_z}{|\vec{p}| - p_z} \right) = \frac{1}{2} \ln \left(\frac{1+\cos\theta}{1-\cos\theta} \right) = -\ln \left[\tan \left(\frac{\theta}{2} \right) \right]$$



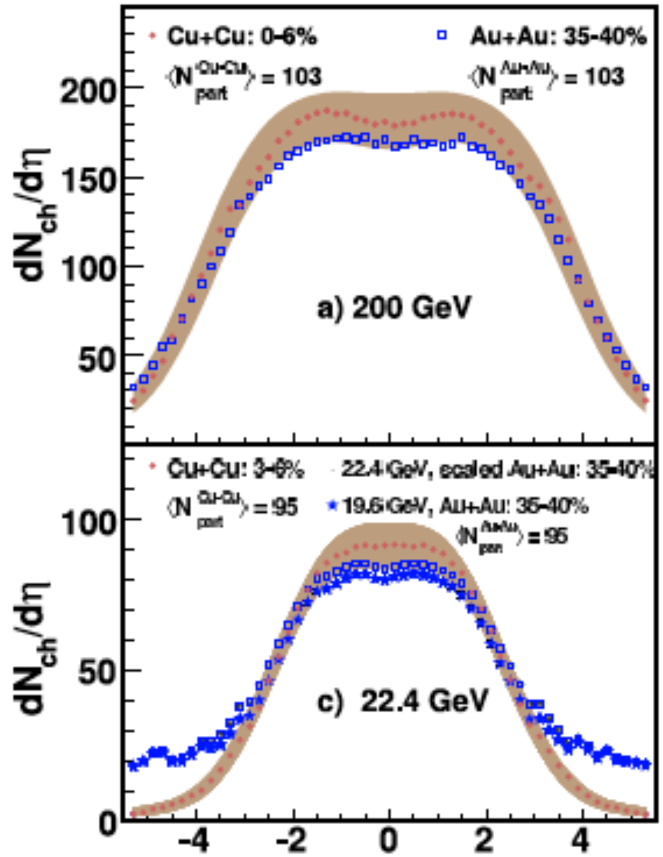
Bjorken's initial estimate of the energy density

1. The energy of particles in one unit of rapidity can be estimated

$$\frac{dE}{d\eta_{\rm pseudo}} \approx \langle E \rangle \times 1.5 \times \frac{dN_{\rm ch}}{d\eta_{\rm pseudo}} \approx 0.5 \text{ GeV} \times 1.5 \times 170 \approx 128 \text{ GeV}$$

2. and this energy that flows into the detector reflects the initial energy in a given space time rapidity slice

$$e_{Bj} \approx \frac{1}{S_\perp}\frac{dE}{dz} \approx \frac{1}{\tau_0 S_\perp}\frac{dE}{d\eta_{\rm pseudo}} \approx \frac{1}{1 \text{ fm/c} \times (5 \text{ fm})^2} \times 128 \text{ GeV} \approx 5 \, \frac{\text{GeV}}{\text{fm}^3}$$



$$\eta_{\rm pseudo} \equiv \frac{1}{2}\ln\left(\frac{|\vec{p}| + p_z}{|\vec{p}| - p_z}\right)$$

Bjorken's hydrodynamical model

1. Assume one-dimensional flow along the beam direction

$$
\begin{aligned}
e &= e(\tau, \eta) \\
u^\mu(t, \vec{x}) &= (u^0(\tau, \eta), 0, 0, u^z(\tau, \eta))
\end{aligned}
$$

2. with the following initial condition

$$
\begin{aligned}
e(\tau_0, \eta) &= e_0 \\
u^\mu(\tau_0, \eta) &= \frac{1}{\tau_0}(t, 0, 0, z) = (\cosh\eta, 0, 0, \sinh\eta)
\end{aligned}
$$

Boost Invariance of initial conditions

1. If we consider a Lorentz boost of initial condition in beam direction

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \cosh\phi & 0 & 0 & \sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix} \qquad v_z = \tanh\phi$$

2. we find the form of the initial conditions are preserved

$$\begin{aligned} \tilde{e}_0 &= e_0 \\ \tilde{u}^{\mu}(\tau_0, \eta) &= u^{\mu}(\tau_0, \eta + \phi) = (\cosh(\eta + \phi), 0, 0, \sinh(\eta + \phi)) \end{aligned}$$

3. so if we have the solution for a fixed η we can find the solution for any η through a longitudinal boost of that solution

τ-η coordinates

1. Now make the coordinate transformation

$$x^\mu = (t, x_\perp, z) \to \tilde{x}^\mu = (\tau, x_\perp, \eta)$$

2. The velocity transforms as $\quad \tilde{u}^\mu = \dfrac{d\tilde{x}^\mu}{dx^\nu} u^\nu$

$$\tilde{u}^\mu = (u^\tau, u^x, u^y, u^\eta) = (1, 0, 0, 0)$$

note: I'll drop the tilde when confusion cannot arise

EoM in τ-η coordinates

1. In this new coordinate system

$$0 = d_\mu T^{\mu\nu} \equiv \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\alpha} T^{\alpha\nu} + \Gamma^\nu_{\mu\alpha} T^{\mu\alpha}$$

2. Exercise: Show that the non-vanishing Christoffel symbols are

$$\Gamma^\eta_{\eta\tau} = \Gamma^\eta_{\tau\eta} = \frac{1}{\tau} \; , \quad \Gamma^\tau_{\eta\eta} = \tau$$

   and that the resulting EoM is

$$\partial_\tau e = -\frac{e + \tau^2 T^{\eta\eta}}{\tau}$$

Euler's equations in τ-η coordinates

1. The ideal stress energy tensor takes the following form

$$\tilde{T}_0^{\mu\nu} = \mathrm{diag}(e, p, p, p/\tau^2)$$

2. And therefore Euler's equation in 1+1 D is

$$\partial_\tau e = -\frac{e+p}{\tau}$$

If you're not a fan of G.R.

1. Exercise: Derive $\partial_\tau e = -\dfrac{e+p}{\tau}$ in flat space

2. You will need

$$T_0^{\mu\nu} = \begin{pmatrix} (e+p)\cosh^2\eta - p & 0 & 0 & (e+p)\cosh\eta\sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ (e+p)\cosh\eta\sinh\eta & 0 & 0 & (e+p)\sinh^2\eta + p \end{pmatrix}$$

and the following derivatives

$$\begin{aligned} \partial_t &= \cosh\eta\partial_\tau - \frac{\sinh\eta}{\tau}\partial_\eta \\ \partial_z &= -\sinh\eta\partial_\tau + \frac{\cosh\eta}{\tau}\partial_\eta \end{aligned}$$

Euler's equation in 1+1 D (cont.)

1. For a massless ideal gas we have the following thermodynamic information

$$\epsilon \propto T^4 \qquad\qquad e = 3p \qquad\qquad s = \frac{e+p}{T}$$

2. and therefore Euler's equation $\partial_\tau e = -\dfrac{e+p}{\tau}$ can be solved

$$e(\tau) = e_0 \left(\frac{\tau_0}{\tau}\right)^{4/3} \qquad T(\tau) = T_0 \left(\frac{\tau_0}{\tau}\right)^{1/3} \qquad s(\tau) = s_0 \left(\frac{\tau_0}{\tau}\right)$$

Three Dimensional Expansion

1. At late times when the longitudinal extent is on the order of the transverse extent

$$\tau \approx R_{Au}/c$$

the evolution becomes three dimensional

$$R \propto \tau \qquad\qquad V \propto \tau^3$$

2. For an ideal expansion the total entropy is constant

$$s(\tau) = s_0 \left(\frac{\tau_0}{\tau}\right)^3$$

$$T(\tau) = T_0 \left(\frac{\tau_0}{\tau}\right)$$

Free Streaming

1. 1D expansion $$\partial_\tau e = -\frac{e + \cancel{\tau^2 T^{\eta\eta}}}{\tau}$$

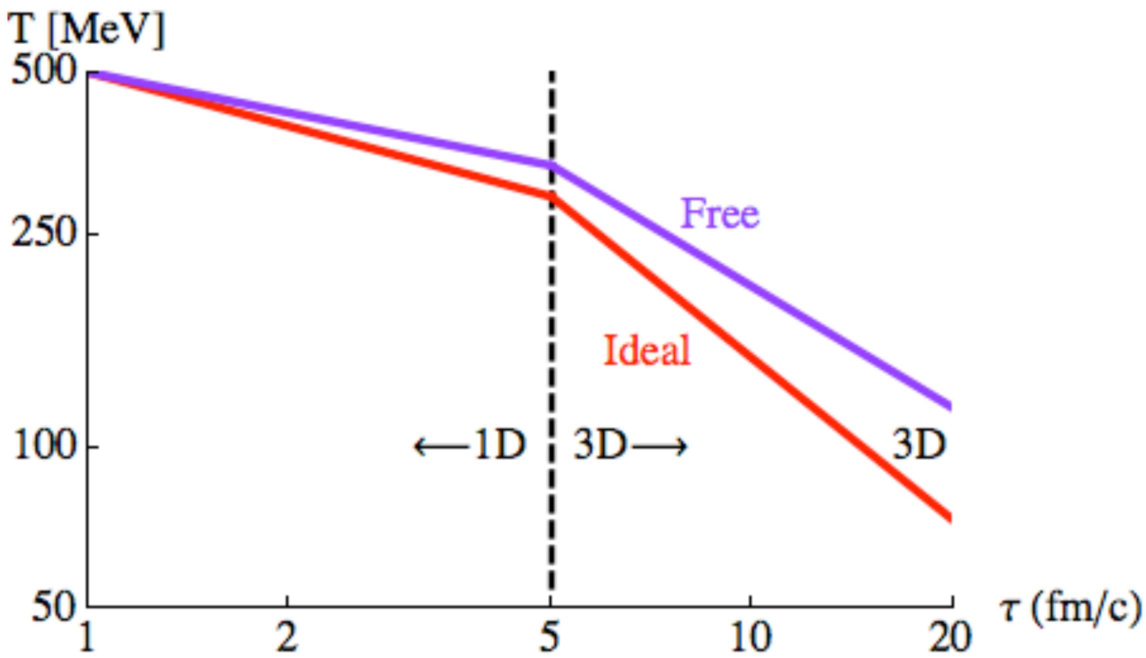
$$e(\tau) = e_0 \left(\frac{\tau_0}{\tau}\right) \qquad T(\tau) = T_0 \left(\frac{\tau_0}{\tau}\right)^{1/4} \qquad s(\tau) = s_0 \left(\frac{\tau_0}{\tau}\right)^{3/4}$$

2. 3D expansion

$$e(\tau) = e_0 \left(\frac{\tau_0}{\tau}\right)^{3} \qquad T(\tau) = T_0 \left(\frac{\tau_0}{\tau}\right)^{3/4} \qquad s(\tau) = s_0 \left(\frac{\tau_0}{\tau}\right)^{9/4}$$

3. In contrast to the Ideal solution (where entropy is conserved) we see here the energy is conserved and the entropy increases with time

Summary



Viscous Hydrodynamics

1. Now lets look at corrections to the ideal stress energy tensor

2. Let us include corrections to first order in gradients of any fields

3. The most general form that these corrections can take are

$$T^{\mu\nu} = T_0^{\mu\nu} - \eta\sigma^{\mu\nu} - \zeta\nabla_\mu u^\mu$$

   where

$$\sigma^{\mu\nu} = \nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3}\Delta^{\mu\nu}\nabla_\lambda u^\lambda$$

4. The above stress tensor with the EoM $\partial_\mu T^{\mu\nu} = 0$ are known as the Navier-Stokes equations.

Viscous Expansion in 0+1 D

1. Exercise: Derive the NS equations for a 0+1 D boost invariant expansion

$$\frac{de}{d\tau} = -\frac{e + p - \frac{4}{3}\frac{\eta}{\tau}}{\tau}$$

2. The gradient expansion is valid when

$$\frac{\eta}{e+p}\frac{1}{\tau} \ll 1$$

Applicability of Hydrodynamics

1. This condition $\dfrac{\eta}{e+p}\dfrac{1}{\tau} \ll 1$ can be written as

$$\underbrace{\frac{\eta}{s}}_{\substack{\text{Medium}\\ \text{Parameter}}} \times \underbrace{\frac{1}{\tau T}}_{\substack{\text{Experimental}\\ \text{Parameter}}} \ll 1$$

2. At RHIC $\dfrac{1}{\tau T} \approx \dfrac{1}{1\ \text{fm} \times 300\ \text{MeV}} \approx 0.66$

   and therefore we need $\dfrac{\eta}{s} \lesssim 0.3$ in order for hydrodynamics to work.

Applicability of Hydrodynamics

1. How does $\dfrac{\eta}{e+p}\dfrac{1}{\tau} \ll 1$ evolve in time?

2. Lets look at two models for the viscosity

- Conformal Gas: $\dfrac{\eta}{s} \sim \text{const.}$ (examples: pQCD, N=4 SYM)

- Constant Cross Section: $\dfrac{\eta}{s} \sim \dfrac{T}{\sigma}$ (example: Boltzmann simulations, HRG)

Conformal Gas

1. For a conformal gas we have $$\frac{\eta}{e+p}\frac{1}{\tau} = \frac{\eta}{s}\frac{1}{\tau T}$$

2. For a 0+1 D longitudinal expansion $$T \sim \begin{cases} \tau^{-1/3} & \text{Ideal Hydro.} \\ \tau^{-1/4} & \text{Free Streaming} \end{cases}$$

$$\frac{\eta}{e+p}\frac{1}{\tau} \sim \frac{1}{\tau^{2/3\cdots 3/4}}$$

3. For a 3D expansion $$T \sim \begin{cases} \tau^{-1} & \text{Ideal Hydro.} \\ \tau^{-3/4} & \text{Free Streaming} \end{cases}$$

$$\frac{\eta}{e+p}\frac{1}{\tau} \sim \frac{1}{\tau^{0\cdots 1/4}}$$

Hard Sphere Gas

1. For a hard sphere gas we have $\dfrac{\eta}{e+p}\dfrac{1}{\tau} = \dfrac{\eta}{s}\dfrac{1}{\tau T} \propto \dfrac{1}{s\sigma\tau} \propto \dfrac{1}{\tau T^3}$

2. For a 0+1 D longitudinal expansion $T \sim \begin{cases} \tau^{-1/3} & \text{Ideal Hydro.} \\ \tau^{-1/4} & \text{Free Streaming} \end{cases}$

$$\frac{\eta}{e+p}\frac{1}{\tau} \sim \frac{1}{\tau^{0\cdots 1/4}}$$

3. For a 3D expansion $T \sim \begin{cases} \tau^{-1} & \text{Ideal Hydro.} \\ \tau^{-3/4} & \text{Free Streaming} \end{cases}$

$$\frac{\eta}{e+p}\frac{1}{\tau} \sim \tau^{2\cdots 5/4}$$

Summary

1. Only for a 3D expansion with constant cross section will the system ever freeze-out

	1D expansion	3D expansion
$\eta \propto T^3$	$\left(\frac{1}{\tau}\right)^{2/3\cdots 3/4}$	$\left(\frac{1}{\tau}\right)^{0\cdots 1/4}$
$\eta \propto \frac{T}{\sigma}$	$\left(\frac{1}{\tau}\right)^{0\cdots 1/4}$	$\tau^{2\cdots 5/4}$